EXTRAPOLATION OF CONTINUOUS LINEAR MULTISTEP METHODS FOR THE NUMERICALSOLUTION OF INITIAL VALUE PROBLEMS FOR ORDINARY DIFFERENTIAL EQUATIONS

BY

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NIGERIA

DECLARATION

I hereby certify that the work which is being presented in this thesis entitled "Extrapolation of Continuous Linear Multistep Methods for the Numerical Solution of Initial Value Problem for Ordinary Differential Equations" in partial fulfillment of the requirements for the award of the degree of Master of Science submitted to the Department of Mathematical Sciences of Nasarawa State University is an authentic record of my own work carried out under the supervision of **Professor Peter Onumanyi**.

The matter embodied in this thesis has not been submitted by me for the award of any other degree.

_____ _____

Egwu, Bernard Anuga **Date**

CERTIFICATION

This Dissertation titled "Extrapolation of Continuous Linear Multistep Methods for the Numerical Solution of Initial Value Problem for Ordinary Differential Equations" meets the regulations governing the award of Masters of Science in Mathematics of the School of Postgraduate Studies Nasarawa State University, Keffi and it is approved for its contribution to knowledge and Literary presentation.

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DEDICATION

This thesis is solely dedicated to God Almighty who has been my help thus far. To Him alone is all the glory.

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ABSTRACT

This dissertation finds approximate solutions of first order ordinary differential equations with initial value conditions using extrapolation of continuous linear multistep methods. By this we mean using the same numerical methods to approximate numerically the solution of an initial value problem (IVP). For example, a problem is to be solved in the interval say [a, b], our interest is to solve the problem / find the solution at a point outside the boundary of the interval to the right say $b + 1(\alpha \epsilon IR)$ without doing any further calculation with those methods. What we want to solve in this problem is using the continuous linear multistep methods for their evaluations outside their range of construction. If the solution is valid within the interval and you are interested in the validity of the solution outside the interval is called extrapolation. Our studies show that the continuous formula can be used to finding values outside the interval of consideration which the discrete formula cannot do. Our studies also compare the results of continuous linear multistep methods and the discrete scheme with the exact solution and discovered that the continuous scheme minimizes the error which is better.

CHAPTER ONE

INTRODUCTION

1.1 Background of Study

An important problem that arises in many scientific and engineering applications is that of approximating limits (intervals) of infinite sequences which in most instances converge very slowly. Thus, to approximate interval with reasonable accuracy, it is necessary to compute a large number of terms, and this is generally costly. These intervals can be approximated economically and with high accuracy by applying suitable extrapolation methods to a small number of terms.

In many problems, such as numerical integration or differential, approximated values for some quantities is based on some step sizes. Ideally, we would like to obtain the limiting value as the step size approaches zero but we cannot in practice take arbitrarily small step sizes because of excessive cost of computation even by the computer. Extrapolation process may help in reducing this cost of additional computations.

In Mathematics, extrapolation is the process of estimating beyond the original observation, range, and the value of a variable on the basis of its relationship with other variables. It is similar to interpolation, which produces estimates between known observations, but extrapolation is subject to greater uncertainty and a higher risk of producing meaningless results. Extrapolation may also mean extension of a method assuming similar methods will be applicable. Extrapolation is applied if the data is assumed to be continuous, smooth, possibly periodic etc.

Extrapolation by multistep methods here uses the Adam-moulton multistep methods of order k=3 to compute the solution of a first order ordinary differential

equation, with initial value conditions. Then we follow the discrete evaluated solution outside the interval of the computed solutions at less cost by the continuous formula.

1.2 Statement of the Problem/Motivation

The extrapolation methods by continuous linear multistep methods is seeking to finding better accuracy of numerical solutions of first order ordinary differential equations with initial value conditions outside the interval of consideration.

1.3 Aims and Objectives

This research work is geared towards achieving more efficient equations with initial value condition using the continuous linear multistep methods and the objectives are:

(a) Extrapolate the interval beyond its consideration for solution of the initial value problem

(b) Compare the results of the continuous scheme and those of the discrete scheme both with the exact solution.

(c) Show that we can obtain values beyond a giving closed interval which is a recipe for solving problems whose data is assumed to be continuous.

1.4. Significance of the Study

At the end of this research work, we shall see that the extrapolation methods by continuous linear multistep methods yield better results faster than the discrete extrapolation methods beyond the interval of solution specified by the numerical problem.

1.5. Scope and Limitation of the Study

The extrapolated method will depend on the accuracy of the continuous linear multistep methods used for non-stiff initial value problems (IVP) mainly.

1.6. Definition of terms

Single- step methods.

The methods of Euler, Heun, Taylor and Runge-Kutta are called single step methods because they use only the information from one previous point to compute the successive point that is only the initial point (x_0, y_0) is used to compute (x_1, y_1) and in general y_n is needed to compute y_{n+1}. Module for Adams-Bashforth-moulton Methods for Ordinary Differential Equation.

Multi-step methods

After several points have been found by using the single step methods it is feasible to use several prior points in the calculation. Multi-step methods use y_{n-3}, y_{n-2}, y_{n-1} and y_n in the computation of y_{n+1}. This method is not self-starting. Four initial points(x_0, y_0), (x_1, y_1), (x_2, y_2) and (x_3, y_3) must be given in advance in order to generate the points $\{((x_n, y_n)\}_{n=4}^{m}$ .A desirable feature of a multi-step method is that the local truncation error (L.T.E) can be determined and a correction term can be included, which improves the accuracy of the answer at each step. Also it is possible to determine if the step size is small enough to obtain accurate value for y_{n+1} , yet large enough so that unnecessary and time consuming calculations are eliminated. If the code for the subroutine is fine tuned then, the combinations of a predictor and corrector requires only two function evaluations of $f(x, y)$ per step.

CHAPTER TWO

LITERATURE REVIEW

Extrapolation is the process of taking data values at points $x_1, \ldots, x_n$ and approximate a value outside the range of the given points. Linear multistep methods are used for the numerical solution of ordinary differential equations. Conceptually, a numerical method starts from an initial point and then takes a short step forward in time to find the next solution point. The process continues with subsequent steps to map out the solution. Single step methods such as Euler's method refer to only one previous point and its derivative to determine the current value. This is called one step method. Another one step method is the Runge-kutta method which takes some intermediate steps (for example, a half-step) to obtain a higher order method, but then discards all previous information before taking a second step. Multistep methods attempt to gain efficiency by keeping and using the information from previous steps rather than discarding it. Consequently, multistep methods refer to several previous points and derivative values. In the case of linear multistep methods, a linear combination of the previous points and derivative values is used.

The existing methods of deriving the linear multistep methods in discrete form include the interpolation approach, numerical integration, Taylor series expansion and through the determination of the order of linear multistep method, continuous collocation and interpolation technique are widely used for the derivation of linear multistep methods. Several continuous linear multistep methods have been derived using different techniques and approaches:

Alabi (2008) derived continuous solvers of initial value problems using Chebyshev polynomial in a multistep collocation technique, Aboiya *etal* (2015): Derived a continuous linear multistep methods using Hermite polynomials as basis function,

Adeniyi *etal* (2008): Derived a Chebyschev-Collocation Approach for Continuous Formulation of Hybrid Methods for IVPs,

Odekunle *etal* (2012): Developed a Continuous Linear Multistep Method using Interpolation and Collocation for the solution of first Order Ordinary Differential Equation with constant Step size,

Awoyemi, (1992): Derived Continuous Linear Multistep Methods for Initial Value Problems,

Awoyemi, (1994): Derived a Sixth Order Linear Multistep Methods for the Direct Solution of a second Order Differential Equations,

Butcher, (1965): Derived a Modified Multistep Methods for Numerical Integration of Ordinary Differential Equations,

Butcher, (2003): Derived a Numerical Methods for Ordinary Differential Equations, Deuflhard,(1983): Derived an Order and Stepsize Control in Extrapolation Methods, James *etal* (2013): Proposed a Continuous Block Method for the solution of second Order Initial Value Problem with constant step size; the method was developed by interpolation and collocation of power series approximate solution,

Onumanyi and Awoyemi (1994): Derived a Sixth Order Linear Multistep Methods for the direct solution of a second Order Differential Equations,

Ehigie *etal* (2010): Derived a Generalised 2-step Continuous Multistep Methods of Hybrid type for the Integration of second Order Ordinary Differential Equations,

Fox and Mayers (1990): Derived a Numerical Solution of Ordinary Differential Equations,

Gear (1971): Numerical Initial Value Problems in Ordinary Differential Equations,

Hall and Watt (1976): Modern Numerical Methods for Ordinary Differential Equations,

Onumanyi *etal* (1994): New Linear Multistep Methods with Continuous Co-efficients for first order Initial Value Problems,

Lambert (1973): Computational Methods in ODEs, Morton, K.W. (1987): Numerical Solution of Ordinary Differential Equations,

Sunday *etal* (2014): A Self Starting Four Step Fifth Order Block Integrator for stiff and Oscillatory Differential Equations,

Prince and Dormond (1989): Practical Runge-Kutta Processes,

Henrici (1962): Discrete Variable Method in Ordinary Differential Equations

Lapidus and Seinfeld (1971): Numerical Solution of Ordinary Differential Equations.

CHAPTER THREE

MATERIALS AND METHODS

3.1 LINEAR MULTI-STEP METHODS (LMMS)

Linear multi-step methods (LMMs) are very popular for solving initial value problems (IVPs) of ordinary differential equations (ODEs). They are also applied to solve higher order ordinary differential equations. Linear multi-step methods are not self starting hence, need starting values from single step methods like Euler method and Runge-kutta family of methods.

The general k-step linear multistep method is as giving in Lambert (1973).

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h \sum_{j=0}^{k} \beta_j f_{n+j} \qquad 3.1$$

Where $\alpha_j$and β_j are uniquely determined and $\alpha_0 + \beta_0 \neq 0, \alpha_k = 1$.

The linear multi-step method in equation (3.1) generates discrete schemes which are used to solve first order ordinary differential equations. Other researchers have introduced the continuous linear multi-step method using the continuous collocations and interpolation approach leading to the development of the continuous linear multi-step methods of the form;

$$y(x) = \sum_{j=0}^{k} \alpha_j(x)\, y_{n+j}(x) + h \sum_{j=0}^{k} \beta_j(x)\, f_{n+j} \qquad 3.2$$

Where $\alpha_j$and β_j are expressed as continuous functions of x and are at least differentiable once.

According to Awoyemi (1992) a continuous form of the Adams-moulton schemes is derived by collocation of the differential system.

3.2 THE CONTINUOUS LINEAR MULTISTEP METHODS

To derive the three-step Adams-Moulton method, we let $n = 4$ in

$$y(x) = \sum_{j=0}^{n} a_j \, h_i(x - x_k) \qquad (3.3)$$

For $x_k \leq x \leq x_{k+p}$ which satisfies an initial value problem of an ordinary differential equation.

Then differentiating 3.3, we obtain:

$$y(x) = a_0 + a_1(x - x_k) + a_2[(x - x_k)^2 - 1] + a_3[(x - x_k)^3 - 3(x - x_k)] + a_4[(x - x_k)^4 - 6(x - x_k)^2 + 3] \qquad (3.4)$$

$$\dot{y}(x) = a_1 + 2a_2(x - x_k) + 3a_3[(x - x_k)^2 - 1] + a_4[4(x - x_k)^3 - 12(x - x_k)] \qquad (3.5)$$

Interpolating equation (3.4) at $x = x_{k+2}$ and collocating equation (3.5) at $x = x_k, x_{k+1}, x_{k+2}, x_{k+3}$ gives:

$$y(x_{k+2}) = a_0 + a_1(x_{k+2} - x_k) + a_2[(x_{k+2} - x_k)^2 - 1] + a_3[(x_{k+2} - x_k)^3 - 3(x_{k+2} - x_k)] + a_4[(x_{k+2} - x_k)^2 - 6(x_{k+2} - x_k)^2 + 3] = y_{k+2}$$

$$\dot{y}(x_k) = a_1 - 3a_3 = f_k$$

$$\dot{y}(x_{k+1}) = a_1 + 2a_2(x_{k+1} - x_k) + 3a_3[(x_{k+1} - x_k)^2 - 1] + a_4[4(x_{k+1} - x_k)^3 - 12(x_{k+1} - x_k)] = f_{k+1}$$

$$\dot{y}(x_{k+2}) = a_1 + 2a_2(x_{k+2} - x_k) + 3a_3[(x_{k+2} - x_k)^2 - 1] + a_4[4(x_{k+2} - x_k)^3 - 12(x_{k+1} - x_k)] = f_{k+2}$$

$$\dot{y}(x_{k+3}) = a_1 + 2a_2(x_{k+3} - x_k) + 3a_3[(x_{k+3} - x_k)^2 - 1]$$

$$+a_4[4(x_{k+3} - x_k)^3 - 12(x_{k+3} - x_k)] = f_{k+3} \qquad (3.6)$$

The equation can be written in matrix form:

$$\begin{matrix} 1 & 2h & (4h^2-1) & (8h^3-6h) & (16h^4-24h^2+3) \\ 0 & h & 0 & -3h & 0 \\ 0 & h & 2h^2 & (3h^3-3h) & (4h^4-12h^2) \\ 0 & h & 4h^2 & (12h^3-3h) & (32h^4-24h^2) \\ 0 & h & 4h^2 & (27h^3-3h) & (108h^4-36h^2) \end{matrix} \begin{matrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{matrix} = \begin{matrix} y_{k+2} \\ hf_k \\ hf_{k+1} \\ hf_{k+2} \\ hf_{k+3} \end{matrix}$$

Solving the system of equations by Gaussian elimination, we have that:

$$a_0 = y_{n+2} + \frac{\mathrm{h}}{12}(2f_{n+3} - 9f_{n+2} + 18f_{n+1} - 11f_n)$$

$$+\frac{1}{8h^3}(f_{n+3} - 3f_{n+2} + 3f_{n+1} - f_n) - \frac{\mathrm{h}}{3}(f_{n+2} + 4f_{n+1} + f_n)$$

$$a_1 = f_n + \frac{1}{2h^2}(-f_{n+3} + 4f_{n+2} - 5f_{n+1} + 2f_n$$

$$a_2 = \frac{1}{12h}(2f_{n+3} - 9f_{n+2} + 18f_{n+1} - 11f_n) +$$

$$\frac{1}{4h^3}(f_{n+3} - 3f_{n+2} + 3f_{n+1} - f_n)$$

$$a_3 = \frac{1}{6h^2}(-f_{n+3} + 4f_{n+2} - 5f_{n+1} + 2f_n)$$

$$a_4 = \frac{1}{24h^3}(f_{n+3} - 3f_{n+2} + 3f_{n+1} - f_n)$$

Substituting for $a_j, j = 0,1,2,3,4$ in equation (3.4) yields the continuous method:

$$y(x) = y_{n+1} - \frac{h}{24}(f_{n+3} - 5f_{n+2} + 19f_{n+1} + 9f_n) + f_n(x - x_n)$$

$$+ \frac{(x - x_n)^2}{12\text{h}}(2f_{n+3} - 9f_{n+2} + 18f_{n+1} - 11f_n)$$

$$+ \frac{(x - x_n)^3}{6h^2}(-f_{n+3} + 4f_{n+2} - 5f_{n+1} + 2f_n)$$

$$+ \frac{(x - x_n)^4}{24h^3}(f_{n+3} - 3f_{n+2} + 3f_{n+1} - f_n)$$

At $n = 0$,

$$y(x) = y_1 - \frac{h}{24}(f_3 - 5f_2 + 19f_1 + 9f_0) + f_0(x - x_0) + \frac{(x-x_0)^2}{12\text{h}}(2f_3 - 9f_2 +$$

$$18f_1 - 11f_0) + \frac{(x-x_0)^3}{6h^2}(-f_3 + 4f_2 - 5f_1 + 2f_0) + \frac{(x-x_0)^4}{24h^3}(f_3 - 3f_2 + 3f_1 - f_0) \quad (3.7)$$

Evaluating equation (3.7) at $x = x_{n+3}$

We obtain the discrete scheme:

$$y_{n+3} = y_{n+2} + \frac{h}{24}(9f_{n+3} + 19f_{n+2} - 5f_{n+1} + f_n) \quad (3.8)$$

Equation (3.7) is the Adam-moulton continuous linear multistep method.

CHAPTER FOUR

RESULTS

Here we attempt to solve first order ordinary differential equations using the continuous linear multistep method as derived in chapter three and compare the results with the discrete adam- moluton scheme.

Example1

Consider the initial value problem

$$\frac{dy}{dx} = -y, 0 \leq x \leq 1, y(0) = 1, \square = 0.1$$

Solution

$$\frac{dy}{dx} = -y, y(0) = 1, \rightarrow y_0 = 1, x_0 = 0, \square = 0.1$$

$$\therefore f(x, y) = -y.$$

By 4th or*der Runge-kutta method, we have*

1st iteration

$k_1 = hf(x_{0,}y_{0)} = hf(0, 1) = 0.1 \times (-1) = -0.1$

$$k_2 = hf(x_0 + \frac{h}{2,}, y_{0+\frac{k1}{2}}) = hf(-y_0 - \frac{k1}{2}) = 0.1\left(-1 + \frac{0.1}{2}\right) = -0.0095.$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k2}{2}\right) = h\left(-y_0 - \frac{k2}{2}\right)$$

$$k_3 = 0.1\left(-1 + \frac{0.095}{2}\right) = -0.09525.$$

$k_4 = hf(x_0 + h, y_0 + k_3) = hf(-y_0 - k_3)$=0.1$(-1 + 0.09525)$.

$$k_4 = -0.09048.$$

$$\boldsymbol{y(0.1) = y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)}$$
$$= 1 + \frac{1}{6}\left((-0.1) + 2(-0.095) + 2(-0.09525) + (-0.09048)\right)$$

$$\boldsymbol{y_1 = 0.904836666667}$$

2nd iteration

$$\boldsymbol{x_0 = 0.1\ , y_1 = 0.904836667}$$

$$k_1 = hf(x_1, y_1) = hf(-y_1) = \ 0.1(-0.90484) = \ -0.090484$$

$$k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right) = hf\left(-y_1 - \frac{k_1}{2}\right) =$$

$$0.1\left(-0.90484 + \frac{0.090484}{2}\right) = 0.1(-0.859598)$$

$$k_2 = -0.0859598$$

$$k_3 = \ hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right) = hf\left(-y_1 - \frac{k_2}{2}\right)$$

$$k_3 = 0.1\left(-0.90484 + \ \frac{0.08596}{2}\right) = 0.1(-0.86182) = -0.086186$$

$$k_4 = hf(x_1 + h, -y_1 - k_3) = hf(-y_1 - k_3)$$

$$k_4 = 0.1(-0.90484 + 0.08619) = -0.081865$$

$$\boldsymbol{y(0.2) = y_2 = y_1 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)}$$

$$y(0.2) = 0.9048366666667 + \frac{1}{6}(-0.090484 +$$

$$2(-0.08596) + 2(-0.08619) - (0.08187))$$

$$y(0.2) = 0.9048366666667 - 0.0861097333333$$

$$\boldsymbol{y(0.2) = 0.818726933333}$$

3rd iteration

$$x_2 = 0.2\ , y_2 = 0.81872693333333$$

$$k_1 = hf(x_2, y_2) = hf(-y_2)$$

$$k_1 = 0.1 \times (-0.8187366) = -0.08187369333$$

$$k_2 = hf\left(x_2 + \frac{h}{2}, y_2 + \frac{k_1}{2}\right) = hf\left(-y_2 - \frac{k_1}{2}\right)$$

$$k_2 = 0.1\left(-0.818736933 + \frac{0.081873}{2}\right) = 0.1(-0.7777935)$$

$$k_2 = -0.07777935$$

$$k_3 = hf\left(x_2 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right) = hf\left(-y_2 - \frac{k_2}{2}\right)$$

$$k_3 = 0.1\left(-0.81873 + \frac{0.07777935}{2}\right)$$

$$k_3 = 0.1(-0.81873 + 0.038889675) = -0.077984032$$

$$k_4 = hf(x_2 + h, -y_2 - k_3) = hf(-y_2 - k_3)$$

$$k_4 = 0.1(-0.81873 + 0.077984032)$$

$$k_4 = -0.0740745968$$

$$\boldsymbol{y(0.3) = y_3 = y_2 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)}$$

$$y(0.3) = 0.81873 + \frac{1}{6}(-0.081873 +$$

$$2(-0.07777935) + 2(-0.077984032) + (-0.077973798)$$

$$y(0.3) = 0.81873 - 0.0779123935$$

$$\boldsymbol{y(0.3) = 0.7408242731}$$

4th iteration

$$x_3 = 0.3\ , y_3 = 0.7408242731$$

$$k_1 = hf(x_3, y_3) = hf(-y_3)$$

$$k_1 = 0.1(-0.7408242731) = -0.07408242731$$

$$k_2 = hf\left(x_3 + \frac{h}{2}, y_3 + \frac{k_1}{2}\right) = hf\left(-y_3 - \frac{k_1}{2}\right)$$

$$k_2 = 0.1(-0.7408242731 + \frac{0.07408242731}{2})$$

$$k_2 = 0.1(-0.7408242731 + 0.0374012137) = -0.0703783059$$

$$k_3 = hf\left(x_3 + \frac{h}{2}, y_3 + \frac{k_2}{2}\right) = hf\left(-y_3 - \frac{k_2}{2}\right)$$

$$k_3 = 0.1(-0.7408242731 + \frac{0.0703783059}{2})$$

$$k_3 = 0.1(-0.7408242731 + 0.035189152) = -0.0705653512$$

$$k_4 = hf(x_3 + h, -y_3 - k_3) = hf(-y_3 - k_3)$$

$$k_4 = 0.1(-0.7408242731 + 0.070563512) = -0.067026076$$

$$\boldsymbol{y(0.4) = y_4 = y_3 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)}$$

$$y(0.4) = 0.7408242731 + \frac{1}{6}(-0.07408242731 + 2(-0.0703783059) + 2(-0.070563512) - 0.067026076)$$

$$y(0.4) = 0.7408242731 - 0.070498689$$

$$\boldsymbol{y(0.4) = 0.670325584}$$

5th iteration

$x_4 = 0.4\ , y_4 = 0.670325584$

$k_1 = hf(x_4, y_4) = hf(-y_4)$

$k_1 = 0.1(-0.670325584) = -0.0670325584$

$$k_2 = hf\left(x_4 + \frac{h}{2}, y_4 + \frac{k_1}{2}\right) = hf\left(-y_4 - \frac{k_1}{2}\right)$$

$$k_2 = 0.1\left(-0.670325584 + \frac{0.067032558}{2}\right)$$

$$= 0.1(-0.670325584 + 0.033516279)$$

$k_2 = -0.06368093$

$$k_3 = hf\left(x_4 + \frac{h}{2}, y_4 + \frac{k_2}{2}\right) = hf\left(-y_4 - \frac{k_2}{2}\right)$$

$k_3 = 0.1\ (-670325584 + 0.031840465) = 0.1(-0.638485118)$

$k_3 = -0.0638485118$

$k_4 = hf(x_4 + h, -y_4 - k_3) = hf(-y_4 - k_3)$

$k_4 = 0.1(-0.670325584 + 0.0638485118) = 0.1(-0.606477073)$

$k_4 = -0.060647707$

$\boldsymbol{y(0.5) = y_5 = y_4 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)}$

$y(0.5) = 0.670325584 + \frac{1}{6}(-0.067032558 +$

$2(-0.06368093) + 2(-0.0638485118) + (-0.060647707))$

$\boldsymbol{y(0.5) = 0.670325584 - 0.063789857 = 0.606535726}$

6th iteration

$x_5 = 0.5\ , y_5 = 0.606535726$

$k_1 = hf(x_5, y_5) = hf(-y_5)$

$k_1 = 0.1\ (-0.606535726) = -0.0606535726$

$k_2 = hf\left(x_5 + \frac{h}{2}, y_5 + \frac{k_1}{2}\right) = hf\left(-y_5 - \frac{k_1}{2}\right)$

$k_2 = 0.1\left(-0.606535726 + \frac{0.0606535726}{2}\right) = 0.1\ (-0.576208939)$

$k_2 = -0.057620893$

$k_3 = hf\left(x_5 + \frac{h}{2}, y_5 + \frac{k_2}{2}\right) = hf\left(-y_5 - \frac{k_2}{2}\right)$

$k_3 = 0.1\left(-0.606535726 + \frac{0.057620893}{2}\right)$

$= 0.1\ (-0.606535726 + 0.028810446)$

$k_3 = -0.057772528$

$k_4 = hf(x_5 + h, -y_5 - k_3) = hf(-y_5 - k_3)$

$k_4 = 0.1(-0.606535726 + 0.057772528) = -0.054876319 \boldsymbol{y(0.6) = y_6}$

$$= \boldsymbol{y_5 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)}$$

$y(0.6) = 0.606535726 + \frac{1}{6}(-0.346316733)$

$\boldsymbol{y(0.6) = 0.606535726 - 0.057719455 = 0.54881627}$

7th iteration

$x_6 = 0.6\ , y_6 = 0.54881627$

$k_1 = hf(x_6, y_6) = hf(-y_6)$

$k_1 = 0.1(-0.54881627) = -0.054881627$

$k_2 = hf\left(x_6 + \frac{h}{2}, y_6 + \frac{k_1}{2}\right) = hf\left(-y_6 - \frac{k_1}{2}\right)$

$k_2 = 0.1(-0.54881627 + 0.027440813) = 0.1(-0.521375456)$

$k_2 = -0.052137545$

$k_3 = hf\left(x_6 + \frac{h}{2}, y_6 + \frac{k_2}{2}\right) = hf\left(-y_6 - \frac{k_2}{2}\right)$

$k_3 = 0.1(-0.54881627 + 0.026068772) = 0.1(-0.522747498)$

$k_3 = -0.052274749$

$k_4 = hf(x_6 + h, -y_6 - k_3) = hf(-y_6 - k_3)$

$k_4 = 0.1(-0.54881627 + 0.052274749) = -0.049654152$

$$\boldsymbol{y(0.7) = y_7 = y_6 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)}$$

$$y(0.7) = 0.54881627 + \frac{1}{6}(0.313360367) =$$

$$0.54881627 - 0.052226727$$

$$\boldsymbol{y(0.7) = 0.496589542}$$

8th iteration

$$x_7 = 0.7\ , y_7 = 0.496589542$$

$$k_1 = hf(x_7, y_7) = hf(-y_7) = 0.1(-0.496589542)$$

$$k_1 = -0.049658954$$

$$k_2 = hf\left(x_7 + \frac{h}{2}, y_7 + \frac{k_1}{2}\right) = hf\left(-y_7 - \frac{k_1}{2}\right)$$

$$k_2 = 0.1(-0.496589542 + 0.024829477)$$

$$k_2 = -0.047176006$$

$$k_3 = hf\left(x_7 + \frac{h}{2}, y_7 + \frac{k_2}{2}\right) = hf\left(-y_7 - \frac{k_2}{2}\right)$$

$$k_3 = 0.1(-0.496589542 + 0.023588003) = -0.047300153$$

$$k_4 = hf(x_7 + h, -y_7 - k_3) = hf(-y_7 - k_3)$$

$$k_4 = 0.1(-0.496589542 + 0.047300153) = -0.044928938$$

$$\boldsymbol{y(0.8) = y_8 = y_7 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)}$$

$$y(0.8) = 0.496589542 + \frac{1}{6}(-0.049658954 + 2(-0.047176006)$$

$+2(-0.047300153)+(-0.044928938))$

$$\boldsymbol{y(0.8) = 0.496589542 - 0.047256701 = 0.44933284}$$

9th iteration

$x_8 = 0.8\ , y_8 = 0.44933284$

$k_1 = hf(x_8, y_8) = hf(-y_8)$

$k_1 = 0.1(-0.44933284) = -0.044933284$

$$k_2 = hf\left(x_8 + \frac{h}{2}, y_8 + \frac{k_1}{2}\right) = hf\left(-y_8 - \frac{k_1}{2}\right)$$

$k_2 = 0.1(-0.44933284 + 0.022466642) = 0.1(-0.426866198)$

$k_2 = -0.042686619$

$$k_3 = hf\left(x_8 + \frac{h}{2}, y_8 + \frac{k_2}{2}\right) = hf\left(-y_8 - \frac{k_2}{2}\right)$$

$k_3 = 0.1(-0.44933284 + 0.021343309) = 0.1(-0.427989531)$

$k_3 = -0.042798953$

$k_4 = hf(x_8 + h, -y_8 - k_3) = hf(-y_8 - k_3)$

$k_3 = 0.1(-0.44933284 + 0.042798953) = 0.1(-0.406533887)$

$k_4 = -0.0406533887$

$$\boldsymbol{y(0.9) = y_9 = y_8 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)}$$

$$y(0.8) = 0.44933284 + \frac{1}{6}(-0.256557816) =$$

$0.44933284 - 0.042759635$

$\boldsymbol{y(0.9) = 0.406573205}$

10th iteration

$x_9 = 0.9\ , y_9 = 0.406573205$

$k_1 = hf(x_9, y_9) = hf(-y_9)$

$k_1 = 0.1(-0.406573205) = -0.0406573205$

$k_2 = hf\left(x_9 + \frac{h}{2}, y_9 + \frac{k_1}{2}\right) = hf\left(-y_9 - \frac{k_1}{2}\right)$

$k_2 = 0.1(-0.406573205 + 0.02032866) = 0.1(-0.0386244544)$

$k_2 = -0.038624454$

$k_3 = hf\left(x_9 + \frac{h}{2}, y_9 + \frac{k_2}{2}\right) = hf\left(-y_9 - \frac{k_2}{2}\right)$

$k_3 = 0.1(-0.406573205 + 0.019312227) = 0.1(-0.387260978)$

$k_3 = -0.038726097$

$k_4 = hf(x_9 + h, -y_9 - k_3) = hf(-y_9 - k_3)$

$k_4 = 0.1(-0.406573205 + 0.038726097) = -0.03678471$

$\boldsymbol{y(1.0) = y_{10} = y_9 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)}$

$y(1.0) = 0.406573205 + \frac{1}{6}(0.232143132) =$

$0.406573205 - 0.038690521$

$$\boldsymbol{y(1.0) = 0.367882684}$$

11th iteration

$$x_{10} = 1.0\ , y_{10} = 0.367882684$$

$$k_1 = hf(x_{10}, y_{10}) = hf(-y_{10}) = 0.1(-0.367882684)$$

$$k_1 = -0.0367882684$$

$$k_2 = \ hf\left(x_{10} + \frac{h}{2}, y_{10} + \frac{k_1}{2}\right) = hf\left(-y_{10} - \frac{k_1}{2}\right)$$

$$k_2 = 0.1(-0.367882684 + 0.018394134) = 0.1(-0.34948854)$$

$$k_2 = \ -0.034948854$$

$$k_3 = \ hf\left(x_{10} + \frac{h}{2}, y_{10} + \frac{k_2}{2}\right) = hf\left(-y_{10} - \frac{k_2}{2}\right)$$

$$k_3 = 0.1(-0.367882684 + 0.017474427) = 0.1(-0.350408257)$$

$$k_3 = -0.035040825$$

$$k_4 = hf(x_{10} + h, -y_{10} - k_3) = hf(-y_{10} - k_3)$$

$$k_4 = 0.1(-0.367882684 + 0.035040825) = 0.1(-0.332841859)$$

$$k_4 = -0.033284185$$

$$\boldsymbol{y(1.1) = y_{11} = y_{10} + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)}$$

$$y(1.1) = 0.367882684 + \frac{1}{6}(0.210051811) =$$

$$0.367882684 - 0.035008635$$

$$y(1.1) = 0.332874049$$

12th iteration

$$x_{11} = 1.1 \ , y_{11} = 0.332874049$$

$$k_1 = hf(x_{11}, y_{11}) = hf(-y_{11}) = 0.1(-0.332874049) = -0.0332874049$$

$$k_2 = hf\left(x_{11} + \frac{h}{2}, y_{11} + \frac{k_1}{2}\right) = 0.1(-0.332874049 + 0.016643702)$$
$$= -0.031623032$$

$$k_3 = hf\left(x_{11} + \frac{h}{2}, y_{11} + \frac{k_2}{2}\right) = 0.1(-0.332874049 + 0.015811517)$$
$$= -0.031706253$$

$$k_4 = hf(x_{11} + h, y_{11} + k_3) = 0.1(-0.332874049 + 0.031706253)$$
$$= -0.030116779$$

$$y_{12} = y(1.2) = 0.332874049 + \frac{1}{6}(-0.0332874049 - 2(0.031623032)$$
$$- 2(0.031706253) - 0.030116779)$$

$$y(1.2) = 0.332874049 + \frac{1}{6}(-0.0332874049 - 0.06326064 - 0.063412506$$
$$- 0.030116779)$$

$$y(1.2) = 0.332874049 - \frac{1}{6}(0.190062753)$$

$$y(1.2) = 0.332874049 - 0.031677125 = 0.301196923$$

Since $\frac{dy}{dx} = -y, \boldsymbol{thus}\ f(x,y) = -y$

$\therefore \quad f(x_0, y_0) = f_0 = -y_0 = -1$

$$f(x_1, y_1) = f_1 = -y_1, thus\ f_1 = -0.9048366667$$

$$f(x_2, y_2) = f_2 = -y_2, thus\ f_2 = -0.8187269333$$

$$f(x_3, y_3) = f_3 = -y_3, thus\ f_3 = -0.7408242731$$

$$f(x_4, y_4) = f_4 = -y_4, thus\ f_4 = -0.670325584$$

$$f(x_5, y_5) = f_5 = -y_5, thus\ f_5 = -0.606535726$$

$$f(x_6, y_6) = f_6 = -y_6, thus\ f_6 = -0.54881627$$

$$f(x_7, y_7) = f_7 = -y_7, thus\ f_7 = -0.496589542$$

$$f(x_8, y_8) = f_8 = -y_8, thus\ f_8 = -0.44933284$$

$$f(x_9, y_9) = f_9 = -y_9, thus\ f_9 = -0.406573205$$

$$f(x_{10}, y_{10}) = f_{10} = -y_{10}, thus\ f_{10} = -0.367882684$$

$$f(x_{11}, y_{11}) = f_{11} = -y_{11}, thus\ f_{11} = -0.332874049$$

Using the continuous scheme

$$\begin{aligned} y(x) = y_{n+1} - \frac{h}{24}(f_{n+3} - 5f_{n+2} + 19f_{n+1} + 9f_n) + f_n(x - x_n) \\ + \frac{(x - x_n)^2}{12h}(2f_{n+3} - 9f_{n+2} + 18f_{n+1} - 11f_n) \\ + \frac{(x - x_n)^3}{6h^2}(-f_{n+3} + 4f_{n+2} - 5f_{n+1} + 2f_n) \\ + \frac{(x - x_n)^4}{24h^3}(f_{n+3} - 3f_{n+2} + 3f_{n+1} - f_n) \end{aligned}$$

At $n = 0$,

$$\begin{aligned} y(x) = y_1 - \frac{h}{24}(f_3 - 5f_2 + 19f_1 + 9f_0) + f_0(x - x_0) \\ + \frac{(x - x_0)^2}{12h}(2f_3 - 9f_2 + 18f_1 - 11f_0) \\ + \frac{(x - x_0)^3}{6h^2}(-f_3 + 4f_2 - 5f_1 + 2f_0) \\ + \frac{(x - x_0)^4}{24h^3}(f_3 - 3f_2 + 3f_1 - f_0) \end{aligned}$$

Thus

$$y(0.1) = 0.9048366667 - \frac{0.1}{24}\big(-0.7408242731 - 5(-0.8187269333) + 19(-0.9048366667) + 9(-1)\big) + (-1)(0.1-0) + \frac{(0.1-0)^2}{12(0.1)}\big(2(-0.7408242731) - 9(-0.8187269333) + 18(-0.9048366667) - 11(-1)\big) + \frac{(0.1-0)^3}{6(0.1)^2}(-(-0.7408242731) + 4(-0.8187269333 - (-0.9048366667) + 2(-1)) + \frac{(0.1-0)^4}{24(0.1)^3}\big((-0.7408242731) - 3(-0.8187269333) + 3(-0.9048366667) - (-1)\big)$$

$$y(0.1) = 0.9048366667 - \frac{0.1}{24}(-22.83908626) - 0.1 + \frac{(0.1-0)^2}{12(0.1)}(0.599833863) +$$

$$\frac{(0.1-0)^3}{6(0.1)^2}(-0.009900129) + \frac{(0.1-0)^4}{24(0.1)^3}(0.000846528)$$

$$y(0.1) = 0.9048366667 + 0.095162859 - 0.1 + 0.004998615 - 0.000165002 + 0.00003527$$

$$\boldsymbol{y(0.1) = 0.904836665.}$$

$$y(0.2) = 0.8187269333 + \frac{0.2}{24}(22.83908626) - 0.2 + \frac{(0.2)^2}{12(0.2)}(0.99833863) - \frac{(0.2)^3}{6(0.2)^2}(0.009900129) + \frac{(0.2)^4}{24(0.2)^3}(0.000846528)$$

$$y(0.2) = 0.8187269333 + 0.190325718 - 0.2 + 0.009997231 - 0.000330004 + 0.00000705$$

$$\boldsymbol{y(0.2) = 0.818726932}$$

$$y(0.3) = 0.7408242731 + \frac{0.3}{24}(22.83908626) - 0.3 + \frac{(0.3)^2}{12(0.3)}(0.99833863) - \frac{(0.3)^3}{6(0.3)^2}(0.009900129) + \frac{(0.3)^4}{24(0.3)^3}(0.000846528)$$

$$y(0.3) = 0.7408242731 + 0.285488578 - 0.3 + 0.014995846 - 0.000495006 + 0.000010581$$

$$\boldsymbol{y(0.3) = 0.740824272}$$

$$y(0.4) = 0.670325584 + \frac{0.4}{24}(22.83908626) - 0.4 +$$

$$\frac{(0.4)^2}{12(0.4)}(0.99833863) - \frac{(0.4)^3}{6(0.4)^2}(0.009900129) +$$

$$\frac{(0.4)^4}{24(0.4)^3}(0.000846528)$$

$$y(0.4) = 0.670325584 + 0.380651437 - 0.4 + 0.019994462 - 0.000660008 + 0.000014108$$

$$\boldsymbol{y(0.4) = 0.670325583}$$

$$y(0.5) = 0.606535726 + \frac{0.5}{24}(22.83908626) - 0.5 +$$

$$\frac{(0.5)^2}{12(0.5)}(0.99833863) - \frac{(0.5)^3}{6(0.5)^2}(0.009900129) +$$

$$\frac{(0.5)^4}{24(0.5)^3}(0.000846528)$$

$$y(0.5) = 0.606535726 + 0.475814297 - 0.5 +$$

$$0.024993077 - 0.00082501 + 0.000017636$$

$$\boldsymbol{y(0.5) = 0.606535726}$$

$$y(0.6) = 0.54881627 + \frac{0.6}{24}(22.83908626) - 0.6 +$$

$$\frac{(0.6)^2}{12(0.6)}(0.99833863) - \frac{(0.6)^3}{6(0.6)^2}(0.009900129) +$$

$$\frac{(0.6)^4}{24(0.6)^3}(0.000846528)$$

$$y(0.6) = 0.54881627 + 0.570977156 - 0.6 + 0.029991693 - 0.000990012 + 0.000021163$$

$$\boldsymbol{y(0.6) = 0.54881627}$$

$$y(0.7) = 0.496589542 + \frac{0.7}{24}(22.83908626) - 0.7 +$$

$$\frac{(0.7)^2}{12(0.7)}(0.99833863) - \frac{(0.7)^3}{6(0.7)^2}(0.009900129) +$$

$$\frac{(0.7)^4}{24(0.7)^3}(0.000846528)$$

$$y(0.7) = 0.496589542 + 0.666140015 - 0.7 +$$

$$0.034990308 - 0.001155015 + 0.00002469$$

$$\boldsymbol{y(0.7) = 0.49658954}$$

$$y(0.8) = 0.44933284 + \frac{0.8}{24}(22.83908626) - 0.8 +$$

$$\frac{(0.8)^2}{12(0.8)}(0.99833863) - \frac{(0.8)^3}{6(0.8)^2}(0.009900129) +$$

$$\frac{(0.8)^4}{24(0.8)^3}(0.000846528)$$

$$y(0.8) = 0.44933284 + 0.761302875 - 0.8 +$$

$$0.039988924 - 0.001320017 + 0.000028217$$

$$\boldsymbol{y(0.8) = 0.449332839}$$

$$y(0.9) = 0.406573205 + \frac{0.9}{24}(22.83908626) - 0.9 +$$

$$\frac{(0.9)^2}{12(0.9)}(0.99833863) - \frac{(0.9)^3}{6(0.9)^2}(0.009900129) +$$

$$\frac{(0.9)^4}{24(0.9)^3}(0.000846528)$$

$$y(0.9) = 0.406573205 + 0.856465734 - 0.9 +$$

$$0.044987539 - 0.001485019 + 0.000031744$$

$$\boldsymbol{y(0.9) = 0.406573203}$$

$$y(1.0) = 0.367882684 + \frac{1.0}{24}(22.83908626) - 1.0 +$$

$$\frac{(1.0)^2}{12(1.0)}(0.99833863) - \frac{(1.0)^3}{6(1.0)^2}(0.009900129) +$$

$$\frac{(1.0)^4}{24(1.0)^3}(0.000846528)$$

$$y(1.0) = 0.367882684 + 0.951628594 - 1 +$$

$$0.049986155 - 0.001650021 + 0.00003527$$

$$\boldsymbol{y(1.0) = 0.367882684}$$

$$y(1.1) = 0.332874049 + \frac{1.1}{24}(22.83908626) - 0.9 +$$

$$\frac{(1.1)^2}{12(1.1)}(0.99833863) - \frac{(1.1)^3}{6(1.1)^2}(0.009900129) +$$

$$\frac{(1.1)^4}{24(1.1)^3}(0.000846528)$$

$$y(1.1) = 0.332874049 + 1.046791454 - 1.1 +$$

$$0.05498477 - 0.001815023 + 0.000038799$$

$$\boldsymbol{y(1.1) = 0.332874049}$$

$$y1.2) = 0.301196923 + \frac{1.2}{24}(22.83908626) - 1.2 + \frac{(1.2)^2}{12(1.2)}(0.599833863) - \frac{(1.2)^3}{6(1.2)^2}(0.009900129) + \frac{(1.2)^4}{24(1.2)^3}(0.000846528)$$

$$y(1.2) = 0.301196923 + 1.141954313 - 1.2 + 0.0599833863 - 0.00919800258 + 0.0000423264$$

$$y(1.2) = 0.301196922$$

EXAMPLE 2

Consider the initial value problem $\frac{dy}{dx} = xy, 0 \leq x \leq 1, y(0) = 1$.

$$\therefore f(x,y) = xy, x_0 = 0, y_0 = 1, h = 0.1$$

By 4th order Runge-Kutta method, we have

1st iteration

$$k_1 = hf(x_{0,}y_{0)} = hf_{(0,\ 1)} = 0.1 \times (0 \times 1) = 0$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = hf\left(0 + \frac{0.1}{2}, 1 + \frac{0}{2}\right) = hf(0.05, 1)$$

$$k_2 = 0.1(0.05 \times 1) = 0.005$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = hf\left(0 + \frac{0.1}{2}, 1 + \frac{0.005}{2}\right)$$

$$= hf(0.05, 1.0025)$$

$$k_3 = 0.1(0.05 \times 1.0025) = 0.0050125$$

$$k_4 = hf(x_0 + h, y_0 + k_3) = 0.1f(0.1, 1 + 0.0050125)$$
$$= 0.1(0.1 \times 1.0050125)$$

$$k_4 = 0.010050125$$

$$\boldsymbol{y(0.1) = y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)}$$

$$y(0.1) = 1 + \frac{1}{6}(0 + 2(0.005) + 2(0.0050125) + 0.0050250625)$$

$$y(0.1) = 1 + \frac{1}{6}(0.01 + 0.0100252 + 0.010050125)$$

$$\boldsymbol{y(0.1) = 1 + \frac{1}{6}(0.030075125) = 1.00501252083333}$$

2nd iteration

$$x_1 = 0.1, \qquad y_1 = 1.00501252083333$$

$$k_1 = hf(x_1, y_1) = 0.1f(0.1, 1.0050125208)$$

$$= 0.1(0.1 \times 1.0050125208)$$

$$k_1 = 0.010050125208$$

$$k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right)$$
$$= hf(0.1 + 0.05, 1.0050125208 + 0.005025062604)$$
$$= hf(0.15, 1.010037583404)$$

$$k_2 = 0.01515056375106$$

$$k_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right)$$
$$= hf(0.1 + 0.05\ , 1.0050125208 + 0.00757528187553)$$

$$k_3 = 0.1(0.15, 1.01258780267553) = 0.01518881704013295$$

$$k_4 = hf(x_1 + h, y_1 + k_3) = 0.1f(0.2\ , 1.0050125208 + 0.015188817)$$

$$k_4 = 0.1(0.2 \times 1.0202013378) = 0.0204040268$$

$$\boldsymbol{y(0.2) = y_2 = y_1 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)}$$

$$y(0.2) = 1.0050125208\ + \frac{1}{6}(0.010050125208\ + 2(0.0151505638)$$
$$+ 2(0.015188817) + 0.0204040268\ )$$

$$y(0.2) = 1.0050125208 + \frac{1}{6}(0.0911329136)$$

$$\boldsymbol{y(0.2) = 1.0050125208 + 0.0151888189 = 1.0202013397}$$

3rd iteration

$$x_2 = 0.2\ , y_2\ = 1.0202013397$$

$$k_1 = hf(x_2, y_2) = 0.1(0.2 \times 1.0202013397) = 0.0204040268$$

$$k_2 = hf\left(x_2 + \frac{h}{2}\ , y_2 + \frac{k_1}{2}\right)$$
$$= 0.1f(0.25, 1.0202013397 + 0.0102020134)$$

$$k_2 = 0.1(0.25 \times 1.0304033531) = 0.0257600838$$

$$k_3 = hf\left(x_2 + \frac{h}{2}, y_2 + \frac{k_2}{2}\right)$$
$$= 0.1f(0.25, 1.0202013397 + 0.0128800419)$$

$$k_3 = 0.1(0.25 \times 1.0330813816) = 0.0258270345$$

$$k_4 = hf(x_2 + h, y_2 + k_3) = 0.1(0.3, 1.0202013397 + 0.0258270345)$$

$$k_4 = 0.1(0.3, 1.0460283742) = 0.0313808512$$

$$\boldsymbol{y(0.3) = y_3 = y_2 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)}$$

$$y(0.3) = 1.0202013397 + \frac{1}{6}(0.0204040268 + 2(0.0257600838)$$
$$+ 2(0.0258270345) + 0.0313808512)$$

$$y(0.3) = 1.0202013397 + \frac{1}{6}(0.1549591146)$$
$$= 1.0202013397 + 0.0258265191$$

$$\boldsymbol{y(0.3) = 1.0460278588}$$

4th iteration

$$x_3 = 0.3, \qquad y_3 = 1.0460278588$$

$$k_1 = hf(x_3, y_3) = 0.1(0.3 \times 1.0460278588) = 0.0313808358$$

$$k_2 = hf\left(x_3 + \frac{h}{2}, y_3 + \frac{k_1}{2}\right)$$
$$= 0.1f(0.35, 1.0460278588 + 0.0156904179)$$

$$k_2 = 0.1(0.35 \times 1.0617182767) = 0.0371601397$$

$$k_3 = hf\left(x_3 + \frac{h}{2}, y_3 + \frac{k_2}{2}\right)$$
$$= 0.1f(0.35, 1.0460278588 + 0.0185800698)$$

$$k_3 = 0.1(0.35 \times 1.0646079286) = 0.0372612775$$

$$k_4 = hf(x_3 + h, y_3 + k_3)$$

$$= 0.1(0.4 \times 1.0460278588 + 0.0372612775)$$

$$k_4 = 0.1(0.4 \times 1.0832891363) = 0.0433315655$$

$$\boldsymbol{y(0.4) = y_4 = y_3 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)}$$

$$y(0.4) = 1.0460278588 + \frac{1}{6}(0.0313808358 + 2(0.0371601397)$$
$$+ 2(0.0372612775) + 0.043315655)$$

$$y(0.4) = 1.0460278588 + \frac{1}{6}(0.223552357)$$
$$= 1.0460278588 + 0.0372587262$$

$$\boldsymbol{y(0.4) = 1.083286585}$$

5th iteration

$$x_4 = 0.4, y_4 = 1.083286585$$

$$k_1 = hf(x_4, y_4) = 0.1(0.4 \times 1.083286585) = 0.0433314634$$

$$k_2 = hf\left(x_4 + \frac{h}{2}, y_4 + \frac{k_1}{2}\right)$$
$$= 0.1(0.45 \times 1.083286585 + 0.0216657317)$$

$$k_2 = 0.1(0.45 \times 1.1049523167) = 0.0497228543$$

$$k_3 = hf\left(x_4 + \frac{h}{2}, y_4 + \frac{k_2}{2}\right)$$
$$= 0.1f(0.45, 1.083286585 + 0.0248614271)$$

$$k_3 = 0.1(0.45 \times 1.1081480136) = 0.0498666606$$

$$k_4 = hf(x_4 + h, y_4 + k_3) =$$

$$0.1f(0.5, 1.083286585 + 0.0498666606)$$

$$k_4 = 0.1(0.5 \times 1.1331532456) = 0.0566576623$$

$$\boldsymbol{y(0.5) = y_5 = y_4 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)}$$

$$y(0.5) = 1.083286585 + \frac{1}{6}(0.0433314634 + 2(0.0497228543)$$
$$+ 2(0.0498666606) + 0.0566576623)$$

$$y(0.5) = 1.083286585 + \frac{1}{6}(0.2991681555)$$
$$= 1.083286585 + 0.0498613592$$

$$\boldsymbol{y(0.5) = 1.1331479442}$$

6th iteration

$$x_5 = 0.5, y_5 = 1.1331479442$$

$$k_1 = hf(x_5, y_5) = 0.1(0.5 \times 1.1331479442) = 0.0566573972$$

$$k_2 = hf\left(x_5 + \frac{h}{2}, y_5 + \frac{k_1}{2}\right)$$
$$= 0.1(0.55 \times 1.1331479442 + 0.0283286986)$$

$$k_2 = 0.1(0.55 \times 1.1614766428) = 0.0638812154$$

$$k_3 = hf\left(x_5 + \frac{h}{2}, y_5 + \frac{k_2}{2}\right)$$
$$= 0.1f(0.55, 1.1331479442 + 0.0319406077)$$

$$k_3 = 0.1(0.55 \times 1.1650885519) = 0.0640798704$$

$$k_4 = hf(x_5 + h, y_5 + k_3) =$$

$$0.1f(0.6, 1.1331479442 + 0.0640798704)$$

$$k_4 = 0.1(0.6 \times 1.1972278146) = 0.0718336689$$

$$\boldsymbol{y(0.6) = y_6 = y_5 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)}$$

$$y(0.6) = 1.1331479442 + \frac{1}{6}(0.0566573972 + 2(0.0638812154)$$
$$+ 2(0.0640798704) + 0.0718336689)$$

$$y(0.6) = 1.1331479442 + \frac{1}{6}(0.3844132377)$$
$$= 1.1331479442 + 0.064068873$$

$$\boldsymbol{y(0.6) = 1.1972168172}$$

7th iteration

$$x_6 = 0.6, y_6 = 1.1972168172$$

$$k_1 = hf(x_6, y_6) = 0.1(0.6 \times 1.1972168172) = 0.071833009$$

$$k_2 = hf\left(x_6 + \frac{h}{2}, y_6 + \frac{k_1}{2}\right)$$
$$= 0.1(0.65 \times 1.1972168172 + 0.0359165045)$$

$$k_2 = 0.1(0.65 \times 1.2331333217) = 0.0801536659$$

$$k_3 = hf\left(x_6 + \frac{h}{2}, y_6 + \frac{k_2}{2}\right)$$
$$= 0.1f(0.65, 1.1972168172 + 0.040076833)$$

$$k_3 = 0.1(0.65 \times 1.2372936502) = 0.0804240873$$

$$k_4 = hf(x_6 + h, y_6 + k_3) =$$

$$0.1f(0.7, 1.1972168172 + 0.080240873)$$

$$k_4 = 0.1(0.7 \times 1.2776409045) = 0.0894348633$$

$$\boldsymbol{y(0.7) = y_7 = y_6 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)}$$

$$y(0.7) = 1.1972168172 + \frac{1}{6}(0.071833009 + 2(0.0801536659)$$
$$+ 2(0.0804240873) + 0.0894348633)$$

$$y(0.7) = 1.1972168172 + \frac{1}{6}(0.4824233787)$$
$$= 1.1972168172 + 0.0804038964$$

$$\boldsymbol{y(0.7) = 1.2776207136}$$

8th iteration

$$x_7 = 0.7, y_7 = 1.2776207136$$

$$k_1 = hf(x_7, y_7) = 0.1(0.7 \times 1.2776207136) = 0.08943345$$

$$k_2 = hf\left(x_7 + \frac{h}{2}, y_7 + \frac{k_1}{2}\right)$$
$$= 0.1(0.75 \times 1.2776207136 + 0.044716725)$$

$$k_2 = 0.1(0.75 \times 1.3223374386) = 0.0991753079$$

$$k_3 = hf\left(x_7 + \frac{h}{2}, y_7 + \frac{k_2}{2}\right)$$
$$= 0.1f(0.75, 1.2776207136 + 0.0495876539)$$

$$k_3 = 0.1(0.75 \times 1.3272083675) = 0.0995406276$$

$$k_4 = hf(x_7 + h, y_7 + k_3) =$$

$$0.1f(0.8, 1.2776207136 + 0.0995406276)$$

$$k_4 = 0.1(0.8 \times 1.3771613412) = 0.1101729073$$

$$\boldsymbol{y(0.8) = y_8 = y_7 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)}$$

$$y(0.8) = 1.2776207136 + \frac{1}{6}(0.08943345 + 2(0.0991753079)$$
$$+ 2(0.0995406276) + 0.1101729073)$$

$$y(0.8) = 1.2776207136 + \frac{1}{6}(0.5970382283)$$
$$= 1.2776207136 + 0.0995063714$$

$$\boldsymbol{y(0.8) = 1.377127085}$$

9th iteration

$$x_8 = 0.8, y_8 = 1.377127085$$

$$k_1 = hf(x_8, y_8) = 0.1(0.8 \times 1.377127085) = 0.1101701668$$

$$k_2 = hf\left(x_8 + \frac{h}{2}, y_8 + \frac{k_1}{2}\right)$$
$$= 0.1(0.85 \times 1.377127085 + 0.0550850834)$$

$$k_2 = 0.1(0.85 \times 1.4322121684) = 0.1217380343$$

$$k_3 = hf\left(x_8 + \frac{h}{2}, y_8 + \frac{k_2}{2}\right)$$
$$= 0.1f(0.85, 1.377127085 + 0.0608690172)$$

$$k_3 = 0.1(0.85 \times 1.4379961022) = 0.1222296687$$

$$k_4 = hf(x_8 + h, y_8 + k_3) =$$

$$0.1f(0.9, 1.377127085 + 0.1222296687)$$

$$k_4 = 0.1(0.9 \times 1.4993567537) = 0.1349421078$$

$$\boldsymbol{y(0.9) = y_9 = y_8 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)}$$

$$y(0.9) = 1.377127085 + \frac{1}{6}(0.1101701668 + 2(0.1217380343)$$
$$+ 2(0.1222296687) + 0.1349421078)$$

$$y(0.9) = 1.377127085 + \frac{1}{6}(0.7330476806)$$
$$= 1.377127085 + 0.1221746134$$

$$\boldsymbol{y(0.9) = 1.4993016984}$$

10th iteration

$$x_9 = 0.9, y_9 = 1.4993016984$$

$$k_1 = hf(x_9, y_9) = 0.1(0.9 \times 1.4993016984) = 0.1349371529$$

$$k_2 = hf\left(x_9 + \frac{h}{2}, y_9 + \frac{k_1}{2}\right) = 0.1(0.95 \times 1.4993016984 + 0.0674685764)$$

$$k_2 = 0.1(0.95 \times 1.5667702748) = 0.1488431761$$

$$k_3 = hf\left(x_9 + \frac{h}{2}, y_9 + \frac{k_2}{2}\right) = 0.1f(0.95, 1.4993016984 + 0.0744215881)$$

$$k_3 = 0.1(0.95 \times 1.5737232865) = 0.1495037122$$

$$k_4 = hf(x_9 + h, y_9 + k_3)$$

$$= 0.1f(1.0, 1.4993016984 + 0.1495037122)$$

$$k_4 = 0.1(1.0 \times 1.6488054106) = 0.1648805411$$

$$\boldsymbol{y(1.0) = y_{10} = y_9 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)}$$

$$y(1.0) = 1.4993016984 + \frac{1}{6}(0.1349371529 + 2(0.1488431761) + 2(0.1495037122) + 0.1648805411)$$

$$y(1.0) = 1.4993016984 + \frac{1}{6}(0.8965114706) = 1.4993016984 + 0.1494185784$$

$$\boldsymbol{y(1.0) = 1.6487202768}$$

11th iteration

$$x_{10} = 1.0, y_{10} = 1.6487202768$$

$$k_1 = hf(x_{10}, y_{10}) = 0.1(1.0 \times 1.6487202768) = 0.1648720277$$

$$k_2 = hf\left(x_{10} + \frac{h}{2}, y_{10} + \frac{k_1}{2}\right) = 0.1(1.05 \times 1.6487202768 + 0.0824360138)$$

$$k_2 = 0.1(1.05 \times 1.7311562906) = 0.1817714105$$

$$k_3 = hf\left(x_{10} + \frac{h}{2}, y_{10} + \frac{k_2}{2}\right) = 0.1f(1.05, 1.6487202768 + 0.0908857053)$$

$$k_3 = 0.1(1.05 \times 1.7396059821) = 0.1826586281$$

$$k_4 = hf(x_{10} + h, y_{10} + k_3) = 0.1f(1.1, 1.6487202768 + 0.1826586281)$$

$$k_4 = 0.1(1.1 \times 1.8313789049) = 0.2014516795$$

$$\boldsymbol{y(1.1) = y_{11} = y_{10} + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)}$$

$$y(1.1) = 1.6487202768 + \frac{1}{6}(0.1648720277 + 2(0.1817714105) + 2(0.1826586281) + 0.2014516795)$$

$$y(1.1) = 1.6487202768 + \frac{1}{6}(1.0951837844) = 1.6487202768 + 0.1825306307$$

$$\boldsymbol{y(1.1) = 1.8312509075}$$

12th iteration

$$x_{11} = 1.1 \ , y_{11} = 1.8312509075$$

$$k_1 = hf(x_{11}, y_{11}) = 0.1(1.1 \times 1.8312509075) = 0.201437599$$

$$k_2 = hf\left(x_{11} + \frac{h}{2}, y_{11} + \frac{k_1}{2}\right) = 0.1(1.15, 1.8312509075 + 0.100718799 = 0.1(1.15, 1.931969707) = 0.222176516$$

$$k_3 = hf\left(x_{11} + \frac{h}{2}, y_{11} + \frac{k_2}{2}\right) = 0.1(1.15, 1.8312509075 + 0.111088258) = 0.223369004$$

$$k_4 = hf(x_{11} + h, y_{11} + k_3) = 0.1(1.2, 1.8312509075 + 0.223369004) = 0.246592913$$

$$y_{12} = y(1.2) = 1.8312509075 + \frac{1}{6}(0.201437599 + 2(0.222176516) + 2(0.223369004) + 0.246592913)$$

$$y(1.2) = 1.8312509075 + \frac{1}{6}(1.339121552)$$

$$y(1.2) = 1.8312509075 + 0.223186925$$

$$y(1.2) = 2.054437832$$

$$\frac{dy}{dx} = xy \qquad 0 \le x \le 1 \quad y(0) = 1 \;\; x_0 = 0, \;\; y_0 = 1$$

$$\therefore f(x,y) = xy$$

$$f(x_0, y_0) = f_0 = x_0 \times y_0 = 0 \times 1 = 0$$

$$f(x_1, y_1) = f_1 = x_1 \times y_1 = 0.1 \times 1.00501252083333$$

$$f_1 = 0.100501252083333$$

$$f(x_2, y_2) = f_2 = x_2 \times y_2 = 0.2 \times 1.0202013397 = 0.2040402679$$

$$f(x_3, y_3) = f_3 = x_3 \times y_3 = 0.3 \times 1.0460278588 = 0.3138083576$$

$$f(x_4, y_4) = f_4 = x_4 \times y_4 = 0.4 \times 1.083286585 = 0.433314634$$

$$f(x_5, y_5) = f_5 = x_5 \times y_5 = 0.5 \times 1.1331479442 = 0.5665739721$$

$$f(x_6, y_6) = f_6 = x_6 \times y_6 = 0.6 \times 1.1972168172 = 0.7183300903$$

$$f(x_7, y_7) = f_7 = x_7 \times y_7 = 0.7 \times 1.2776207136 = 0.8943344995$$

$$f(x_8, y_8) = f_8 = x_8 \times y_8 = 0.8 \times 1.377127085 = 1.101701668$$

$$f(x_9, y_9) = f_9 = x_9 \times y_9 = 0.9 \times 1.4993016984 = 1.3493715286$$

$$f(x_{10}, y_{10}) = f_{10} = x_{10} \times y_{10} = 1.0 \times 1.6487202768 = 1.6487202768$$

$$f(x_{11}, y_{11}) = f_{11} = x_{11} \times y_{11} = 1.1 \times 1.8312509075 = 2.0143759982$$

Using the continuous scheme

$$y(x) = y_{n+1} - \frac{h}{24}(f_{n+3} - 5f_{n+2} + 19f_{n+1} + 9f_n) + f_n(x - x_n) + \frac{(x - x_n)^2}{12h}(2f_{n+3} - 9f_{n+2} + 18f_{n+1} - 11f_n) + \frac{(x - x_n)^3}{6h^2}(-f_{n+3} + 4f_{n+2} - 5f_{n+1} + 2f_n) + \frac{(x - x_n)^4}{24h^3}(f_{n+3} - 3f_{n+2} + 3f_{n+1} - f_n)$$

At $n = 0$, we have:

$$y(x) = y_1 - \frac{h}{24}(f_3 - 5f_2 + 19f_1 + 9f_0) + f_0(x - x_0) + \frac{(x - x_0)^2}{12h}(2f_3 - 9f_2 + 18f_1 - 11f_0) + \frac{(x - x_0)^3}{6h^2}(-f_3 + 4f_2 - 5f_1 + 2f_0) + \frac{(x - x_0)^4}{24h^3}(f_3 - 3f_2 + 3f_1 - f_0)$$

Solving the initial value problem $\frac{dy}{dx} = xy,\ 0 \le x \le 1,\ y(0) = 1, h = 0.1$

$$y(0.1) = 1.0050125208 - \frac{h}{24}(0.3138083576 - 5(0.2040402679) + 19(0.10050125208) + 9 \times 0) + \frac{(x - x_0)^2}{12h}(2(0.3138083576) - 9(0.2040402679) + 18(0.10050125208) - 11 \times 0) + \frac{(x - x_0)^3}{6h^2}(-(0.3138083576) + 4(0.2040402679) - 5(0.10050125208) + 2 \times 0) +$$

$$\frac{(x - x_0)^4}{24h^3}(0.3138083576 - 3(0.2040402679) + 3(0.10050125208) - 0)$$

$$y(0.1) = 1.0050125208 - \frac{h}{24}(1.2031308076) + \frac{(x - x_0)^2}{12h}(0.6002768415) + \frac{(x - x_0)^3}{6h^2}(-0.0001535464) + \frac{(x - x_0)^4}{24h^3}(0.0031913101)$$

$$y(0.1) = 1.0050125208 - \frac{0.1}{24}(1.2031308076) + \frac{(0.1-0)^2}{12(0.1)}(0.6002768415) + \frac{(0.1-0)^3}{6(0.1)^2}(-0.0001535464) + \frac{(0.1-0)^4}{24(0.1)^3}(0.0031913101)$$

$$y(0.1) = 1.0050125208 - 0.005013045 + 0.005002307 - 0.00000255910667 + 0.0000132971254$$

$$\boldsymbol{y(0.1) = 1.0050125208}$$

$$y(0.2) = 1.022013397 - \frac{0.2}{24}(1.2031308076) + \frac{(0.2)^2}{12(0.2)}(0.6002768415) + \frac{(0.2)^3}{6(0.2)^2}(-0.0001535464) + \frac{(0.2)^4}{24(0.2)^3}(0.0031913101)$$

$$y(0.2) = 1.0202013397 - 0.0100260901 + 0.010004614 - 0.00000511821333 + 0.0000265942508$$

$$\boldsymbol{y(0.2) = 1.0202013396}$$

$$y(0.3) = 1.0460278588 - \frac{0.3}{24}(1.2031308076) + \frac{(0.3)^2}{12(0.3)}(0.6002768415) + \frac{(0.3)^3}{6(0.3)^2}(-0.0001535464) + \frac{(0.3)^4}{24(0.3)^3}(0.0031913101)$$

$$y(0.3) = 1.0460278588 - 0.0150391351 + 0.015006921 - 0.00000767732 + 0.0000398913762$$

$$\boldsymbol{y(0.3) = 1.0460278588}$$

$$y(0.4) = 1.083286585 - \frac{0.4}{24}(1.2031308076) + \frac{(0.4)^2}{12(0.4)}(0.6002768415) + \frac{(0.4)^3}{6(0.4)^2}(-0.0001535464) + \frac{(0.4)^4}{24(0.4)^3}(0.0031913101)$$

$$y(0.4) = 1.083286585 - 0.0200521801 + 0.020009228 - 0.0000102364267 + 0.0000531885017$$

$$\boldsymbol{y(0.4) = 1.083286585}$$

$$y(0.5) = 1.1331479442 - \frac{0.5}{24}(1.2031308076) + \frac{(0.5)^2}{12(0.5)}(0.6002768415) + \frac{(0.5)^3}{6(0.5)^2}(-0.0001535464) + \frac{(0.5)^4}{24(0.5)^3}(0.0031913101)$$

$$y(0.5) = 1.1331479442 - 0.0250652252 + 0.0250115351 - 0.0000127955333 + 0.0000664856271$$

$$\boldsymbol{y(0.5) = 1.1331479442}$$

$$y(0.6) = 1.1972168172 - \frac{0.6}{24}(1.2031308076) + \frac{(0.6)^2}{12(0.6)}(0.6002768415) + \frac{(0.6)^3}{6(0.6)^2}(-0.0001535464) + \frac{(0.6)^4}{24(0.6)^3}(0.0031913101)$$

$$y(0.6) = 1.1972168172 - 0.0300782702 + 0.0300138421 - 0.00001535464 + 0.0000797827525$$

$$\boldsymbol{y(0.6) = 1.1972168172}$$

$$y(0.7) = 1.2776207136 - \frac{0.7}{24}(1.2031308076) + \frac{(0.7)^2}{12(0.7)}(0.6002768415) + \frac{(0.7)^3}{6(0.7)^2}(-0.0001535464) + \frac{(0.7)^4}{24(0.7)^3}(0.0031913101)$$

$$y(0.7) = 1.2776207136 - 0.0350913152 + 0.0350161491 - 0.0000179137467 + 0.0000930798779$$

$$\boldsymbol{y(0.7) = 1.2776207136}$$

$$y(0.8) = 1.377127085 - \frac{0.8}{24}(1.2031308076) + \frac{(0.8)^2}{12(0.8)}(0.6002768415) + \frac{(0.8)^3}{6(0.8)^2}(-0.0001535464) + \frac{(0.8)^4}{24(0.8)^3}(0.0031913101)$$

$$y(0.8) = 1.377127085 - 0.0401043603 + 0.0400184561 - 0.0000204728533 + 0.000106377003$$

$$\boldsymbol{y(0.8) = 1.3771270849}$$

$$y(0.9) = 1.4993016984 - \frac{0.9}{24}(1.2031308076) + \frac{(0.9)^2}{12(0.9)}(0.6002768415) + \frac{(0.9)^3}{6(0.9)^2}(-0.0001535464) + \frac{(0.9)^4}{24(0.9)^3}(0.0031913101)$$

$$y(0.9) = 1.4993016984 - 0.0451174053 + 0.0450207631 - 0.00002303196 + 0.000119674129$$

$$\boldsymbol{y(0.9) = 1.4993016984}$$

$$y(1.0) = 1.6487202768 - \frac{1.0}{24}(1.2031308076) + \frac{(1.0)^2}{12(1.0)}(0.6002768415) + \frac{(1.0)^3}{6(1.0)^2}(-0.0001535464) + \frac{(1.0)^4}{24(1.0)^3}(0.0031913101)$$

$$y(1.0) = 1.6487202768 - 0.0501304503 + 0.0500230701 - 0.0000255910667 + 0.000132971254$$

$$\boldsymbol{y(1.0) = 1.6487202768}$$

$$y(1.1) = 1.8312509075 - \frac{1.1}{24}(1.2031308076) + \frac{(1.1)^2}{12(1.1)}(0.6002768415) + \frac{(1.1)^3}{6(1.1)^2}(-0.0001535464) + \frac{(1.1)^4}{24(1.1)^3}(0.0031913101)$$

$$y(1.1) = 1.8312509075 - 0.0551434953 + 0.0550253771 - 0.0000281501733 + 0.00014626838$$

$$\boldsymbol{y(1.1) = 1.8312509075}$$

$$y1.2) = 2.054437832 - \frac{1.2}{24}(1.2031308076) + \frac{(1.2)^2}{12(1.2)}(0.6002768415) - \frac{(1.2)^3}{6(1.2)^2}(0.0001535464)s + \frac{(1.2)^4}{24(1.2)^3}(0.0031913101)$$

$$y(1.2) = 2.054437832 - 0.06015654 + 0.06002784 - 0.00003070928 + 0.000159565505$$

$$\boldsymbol{y(1.2) = 2.054437832}$$

Example 3

Consider the IVP $\frac{dy}{dx} = -2xy^2$ **,** $y(0) = 1, h = 0.2$

Exact solution:

$$y(x) = \frac{1}{x^2 + 1}$$

Using Runge-Kutta method of order 4, with step length $h = 0.2$,

$k_1 = -0.0$

$k_2 = -0.040000001192092904$

$k_3 = -0.03841600109815598$

$k_4 = -0.073971505016004751$

$y_1(0.2) = 0.9615327483765758$

2nd iteration

$k_1 = -0.7396362030033652$

$k_2 = -0.102575334202282$

$k_3 = -0.09942553577510745$

$k_4 = -0.11891661890710704$

$y_2(0.4) = 0.8620524180696251$

3rd iteration

$k_1 = -0.11890150298349086$

$k_2 = -0.12883389087826705$

$k_3 = -0.12724447323187424$

$k_4 = -0.12958625565317425$

$\boldsymbol{y_3(0.6) = 0.7352783369268004}$

4th iteration

$k_1 = -0.12975221972783935$

$k_2 = -0.12584296464465622$

$k_3 = -0.1265778509537273$

$k_4 = -0.11856521365315237$

$\boldsymbol{y_4(0.8) = 0.6097518261638406}$

5th iteration

$k_1 = -0.118975136188959$

$k_2 = -0.10900467458369312$

$k_3 = -0.11098872146906143$

$k_4 = -0.09950585680741567$

$\boldsymbol{y_5(1.0) = 0.500071953135232}$

6th iteration

$k_1 = -0.1000287833$

$k_2 = -0.0891227958$

$k_3 = -0.091295541$

$k_4 = -0.0802071144$

$y(1.2) = 0.4098931913$

7th iteration

$\boldsymbol{x_6 = 1.2, y_6 = 0.4098931913}$

$k_1 = hf(x_6, y_6) = 0.2(-2(1.2)(0.4098931913)^2 = -0.08045965$

$k_2 = hf\left(x_6 + \frac{h}{2}, y_6 + \frac{k_1}{2}\right) = 0.2f(1.3, 0.4098931913 - 0.040322982$

$k_2 = 0.2f(1.3, 0.369570209) = 0.2(-2(1.3)(0.369570209)^2 = -0.071022712$

$k_3 = hf\left(x_6 + \frac{h}{2}, y_6 + \frac{k_2}{2}\right) = 0.2f(1.3, 0.4098931913 - 0.035511356$

$k_3 = 0.2f(1.3, 0.374381835) = 0.2(-2(1.3)(0.374381835)^2 = -0.072884114$

$k_4 = hf(x_6 + h, y_6 + k_3) = 0.2f(1.4, 0.4098931913 - 0.072884114)$

$k_4 = 0.2(-2(1.4)(0.337009077)^2 = -0.063602066$

$$y_7 = y(1.4) = 0.4098931913 + \frac{1}{6}(-0.080645965 - 2(0.071022712) - 2(0.072884114) - 0.063602066)$$

$y(1.4) = 0.4098931913 - \frac{1}{6}(0.432061683) = 0.4098931913 - 0.07201028$

$\boldsymbol{y(1.4) = 0.337882911}$

Now

$\boldsymbol{f(x, y) = -2xy^2}$

Thus

$$f_0(x_0,y_0) = f_0(-2 \times x_0 \times {y_0}^2) = 0$$

$$f_1(x_1,y_1) = f_1(-2 \times x_1 \times {y_1}^2) = -2(0.2)(0.9615327483765758)^2$$
$$= -0.3698180904802445727$$

$$f_2(x_2,y_2) = f_2(-2 \times x_2 \times {y_2}^2) = -2(0.4)(0.8620524180696251)^2$$
$$= -0.5945074971997501564$$

$$f_3(x_3,y_3) = f_3(-2 \times x_3 \times {y_3}^2) = -2(0.6)(0.735278336926268)^2$$
$$= -0.64876107930460969043$$

$$f_4(x_4,y_4) = f_4(-2 \times x_4 \times {y_4}^2) = -2(0.8)(0.6097518261638406)^2$$
$$= -0.5948756632162215777$$

$$f_5(x_5,y_5) = f_5(-2 \times x_5 \times {y_5}^2) = -2(1.0)(0.500071953135)^2$$
$$= -0.50001439166249673388$$

Continuous solution

$$\boldsymbol{y(x) = y_1 - \frac{h}{24}(f_3 - 5f_2 + 19f_1 + 9f_0) + f_0(x - x_0)}$$
$$\boldsymbol{+ \frac{(x - x_0)^2}{12h}(2f_3 - 9f_2 + 18f_1 - 11f_0)}$$
$$\boldsymbol{+ \frac{(x - x_0)^3}{6h^2}(-f_3 + 4f_2 - 5f_1 + 2f_0)}$$
$$\boldsymbol{+ \frac{(x - x_0)^4}{24h^3}(f_3 - 3f_2 + 3f_1 - f_0)}$$

$$y(0.2) = 0.9615327483765758$$
$$- \frac{0.2}{24}(-0.6487610793 - 5(-0.5945074972) + 19(-0.3698180904$$
$$+ 0) + 0$$
$$+ \frac{(0.2 - 0)^2}{12(0.2)}(2(-0.6487610793) - 9(-0.5945074972)$$
$$+ 18(-0.3698180904) - 0)$$
$$+ \frac{(0.2 - 0)^3}{6(0.2)^2}(-(-0.6487610793) + 4(-0.5945074972)$$
$$- 5(-0.3698180904) + 0)$$
$$+ \frac{(0.2 - 0)^4}{24(0.2^)^3}(-0.6487610793 - 3(-0.5945074972)$$
$$+ 3(-0.3698180904) - 0)$$

$$y(0.2) = 0.9615327483765758 - \frac{0.2}{24}(-4.7027673109) + 0 + \frac{(0.2-0)^2}{12(0.2)}(-2.603680311) + \frac{(0.2-0)^3}{6(0.2)^2}(0.1198215425) + \frac{(0.2-0)^4}{24(0.2)^3}(0.0253071411)$$

$$y(0.2) = 0.9615327483765758 + 0.0391897276 - 0.0433946718 + 0.0039940514 + 0.00210892842$$

$$\boldsymbol{y(0.2) = 0.9615327484}$$

$$y(0.4) = 0.8620524180696251 - \frac{0.4}{24}(-4.7027673109) + 0 + \frac{(0.4-0)^2}{12(0.4)}(-2.603680311) + \frac{(0.4-0)^3}{6(0.4)^2}(0.1198215425) + \frac{(0.4-0)^4}{24(0.4)^3}(0.0253071411)$$

$$y(0.4) = 0.8620524180696251 + 0.078379455 - 0.086789343 + 0.007988102833 + 0.000421785685$$

$$\boldsymbol{y(0.4) = 0.862052418}$$

$$y(0.6) = 0.7352783369268004 - \frac{0.6}{24}(-4.7027673109) + 0 + \frac{(0.6)^2}{12(0.6)}(-2.603680311) + \frac{(0.6)^3}{6(0.6)^2}(0.1198215425) + \frac{(0.6)^4}{24(0.6)^3}(0.0253071411)y(0.6)$$

$$= 0.7352783369268004 + 0.117569182 - 0.130184015 + 0.011982154 + 0.0006326785275\boldsymbol{y(0.6) = 0.735278336}$$

$$y(0.8) = 0.6097518261638406 - \frac{0.8}{24}(-4.7027673109) + 0 + \frac{(0.8)^2}{12(0.8)}(-2.603680311) + \frac{(0.8)^3}{6(0.8)^2}(0.1198215425) + \frac{(0.8)^4}{24(0.8)^3}(0.0253071411)$$

$$y(0.8) = 0.6097518261638406 + 0.15675891 - 0.173578687 + 0.015976205 + 0.00084357137$$

$$\boldsymbol{y(0.8) = 0.609751825}$$

$$y(1.0) = 0.500071953135232 - \frac{1.0}{24}(-4.7027673109) + 0 + \frac{(1.0)^2}{12(1.0)}(-2.603680311) + \frac{(1.0)^3}{6(1.0)^2}(0.1198215425) + \frac{(1.0)^4}{24(1.0)^3}(0.0253071411)$$

$$y(1.0) = 0.500071953135232 + 0.195948637 - 0.216973359 + 0.019970257 + 0.001054464212$$

$$\boldsymbol{y(1.0) = 0.500071952}$$

$$y(1.2) = 0.4098931913 - \frac{1.2}{24}(-4.7027673109) + 0 + \frac{(1.2)^2}{12(1.2)}(-2.603680311) + \frac{(1.2)^3}{6(1.2)^2}(0.1198215425) + \frac{(1.2)^4}{24(1.2)^3}(0.0253071411)$$

$$y(1.2) = 0.4098931913 + 0.2351383655 - 0.2603680311 + 0.0239643085 + 0.0012653571$$

$$\boldsymbol{y(1.2) = 0.4098931913}$$

$$\boldsymbol{y(1.4)} = 0.337882911 + \frac{1.4}{24}(4.7027673109) + \frac{(1.4)^2}{12(1.4)}(-2.603680311) + \frac{(1.4)^3}{6(1.4)^2}(0.1198215425) + \frac{(1.4)^4}{24(1.4)^3}(0.0253071411)$$

$$y(1.4) = 0.337882911 + 0.274328093 - 0.303762702 + 0.027958359 + 0.001476249897$$

$$\boldsymbol{y(1.4) = 0.33788291}$$

Table1. Result of example1, with step size h = 0.1

x-value	Exact solution $y(x)$	Discrete Adams-moulton Approximation of $y_n(x)$	Continuous Adams-Moulton Approximation of $y_n(x)$
0.1	0.9048374180	0.9048375000	0.9048366652
0.2	0.8187307531	0.8187309014	0.8187269321
0.3	0.7408182207	0.7408181288	0.7408242720
0.4	0.6703200460	0.6703196483	0.6703255831
0.5	0.6065306597	0.6065299217	0.6065357260
0.6	0.5488116361	0.5488105242	0.5488162700
0.7	0.4965853038	0.4965838086	0.4965895400
0.8	0.4493289641	0.4493270829	0.4493328390
0.9	0.4065696597	0.4065674014	0.4065732030
1.0	0.3678794412	0.3678768199	0.3678826842
1.1	0.332871083		0.3328740490
1.2	0.301194211		0.301196923

From the results on the table above, it brings to bear the fact that the continuous scheme agrees in precision with the exact solution up to the fourth decimal place and as we take the step further it is still correct. Therefore by induction, we can conclude that the extrapolation method by continuous linear multistep method can be used to extrapolate an interval or a boundary to any length and this is common to all the three examples in this thesis.

Table2. Result of example 2, with step size h=0.1

x-value	Exact solution $y(x)$	Discrete Adams-moulton Approximation of $y_n(x)$	Continuous Adams-Moulton Approximation of $y_n(x)$
0.1	1.0050125208	1.0050125208	1.0050125208
0.2	1.0202013400	1.0202013398	1.0202013396
0.3	1.0460278599	1.0460285650	1.0460278588
0.4	1.0832870676	1.0832888045	1.0832865851
0.5	1.1331484530	1.1331514299	1.1331479442
0.6	1.1972173631	1.1972215971	1.1972168172
0.7	1.2776213132	1.2776264969	1.2776207136
0.8	1.3771277643	1.3771330553	1.3771270849
0.9	1.4993025001	1.4993061863	1.4993016984
1.0	1.6487212707	1.6487202423	1.6487202768
1.1	1.8312522088		1.8321509075
1.2	2.054433211		2.0544378320

Table3. Result of example3, with step size h=0.2

x-value	Exact solution $y(x)$	Discrete Adams-moulton Approximation of $y_n(x)$	Continuous Adams-Moulton Approximation of $y_n(x)$
0.2	0.961538461	0.9615327484	0.9615327484
0.4	0.862068965	0.8620524181	0.8620524180
0.6	0.735294117	0.7346740700	0.7352784336
0.8	0.609756097	0.6096314860	0.6097518250
1.0	0.500000000	0.5308761500	0.5000071952
1.2	0.409836065		0.4098931913
1.4	0.337882911		0.3378829210

CHAPTER FIVE

DISCUSSION, CONCLUSION AND RECOMMENDATION

5.1 Discussion

In chapter three continuous and discrete linear multistep methods were derived through the technique of collocation and interpolation using the probabilists' Hermite polynomials as basis function. The result has shown that continuous and discrete linear multistep methods can be derived using polynomial function and approach. The results are shown on the table (50-52). From the results obtained it shows that the continuous scheme produces almost the same result as that of Adam-moulton but little precision than Adam-moulton methods. The result has also shown that we can extrapolate using the continuous scheme which will help in solving problems that are continuous, something the existing discrete schemes cannot do.

In addition, this research beamed its search light on how to use the continuous linear multi step method to obtain values outside an interval of consideration which means that we tried to extrapolate the continuous linear multistep interval beyond certain interval to obtain values that can be compared with their corresponding exact solutions. Runge –Kutta of 4^{th} order is used to obtain past points for the continuous scheme.

5.2 Conclusion

The continuous scheme is used to obtain approximate values of ordinary differential equations with initial value problems. Extrapolation of Continuous Linear Multistep Method is efficient for the numerical solution of initial value problems of ordinary differential equations. The method is safer and shows with clarity the processes involved at reaching the answer. Based on the results we have obtained from this

research work, any linear and non-linear Initial Value Problem of Ordinary Differential Equation that has exact solution can be solved with this method.

5.3 RECOMMENDATION

Considering the results discussed in pages50-52, we make some useful recommendations from this dissertation as follows:

i. The continuous linear multistep methods can be used to numerically solve initial value problems of ordinary differential equations

ii. Any problem both in mathematics and engineering whose data is assumed to be continuous can be solved with the help of the continuous scheme

5.4 CONTRIBUTION TO KNOWLEDGE

This dissertation brings to bare the fact that; any ordinary differential equation with initial value problem can be solved numerically with the help of the continuous linear multistep methods. Also, this dissertation can be improved upon if K(number of steps) is increased for higher order. We have been able to establish that an interval/boundary can continuously be extrapolated which we showed with three numerical examples including and non-linear Initial Value Problem.

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